Homogeneous Linear Systems

Definition: A $m \times n$ linear system

if all b's = ∅

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \qquad \vdots \qquad\qquad\qquad\qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases} \tag{1}$$

is called homogeneous if $b_1 = b_2 = \cdots = b_m = 0$.

Concept: All homogeneous linear systems have the trivial solution

$$x_1 = 0, \qquad x_2 = 0, \qquad \cdots, \qquad x_n = 0$$

and are therefore consistent.

Example 8: Show that the underdetermined homogeneous linear system with the given augmented matrix has infinitely many solutions.

$$\left[\begin{array}{cccc|c} 1 & 3 & 1 & 0 & 0 \\ 3 & -1 & -3 & -4 & 0 \\ 2 & 1 & -3 & 0 & 0 \end{array}\right] \tag{2}$$

$$\left[\begin{array}{cccc|c} 1 & 3 & 1 & 0 & 0 \\ 3 & -1 & -3 & -4 & 0 \\ 2 & 1 & -3 & 0 & 0 \end{array}\right] \underset{R_2 := R_2 - 3R_1}{\overset{R_3 := R_3 - 2R_1}{\sim}} \left[\begin{array}{cccc|c} 1 & 3 & 1 & 0 & 0 \\ 0 & -10 & -6 & -4 & 0 \\ 0 & -5 & -5 & 0 & 0 \end{array}\right] \underset{R_2 \leftrightarrow R_3}{\overset{R_2 := -\frac{1}{5}R_2}{\sim}} \left[\begin{array}{cccc|c} 1 & 3 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & -10 & -6 & -4 & 0 \end{array}\right] \sim$$

$$\overset{R_3 := R_3 + 10R_2}{\sim} \left[\begin{array}{cccc|c} 1 & 3 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 4 & -4 & 0 \end{array}\right] \underset{R_3 := \frac{1}{4}R_3}{\overset{R_2 := R_2 - R_3}{\sim}} \left[\begin{array}{cccc|c} 1 & 3 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right] \overset{R_1 := R_1 - 3R_2}{\sim}$$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & -2 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right]$$

$x_4 = t$ - free var

$x_3 - t = 0 \Rightarrow x_3 = t$

$x_2 + t = 0 \Rightarrow x_2 = -t$

$x_1 - 2t = 0 \Rightarrow x_1 = 2t$

$$\boxed{\begin{aligned} x_1 &= 2t \\ x_2 &= -t \\ x_3 &= t \\ x_4 &= t \end{aligned}} \qquad t \in \mathbb{R}.$$

> *Theorem 3 (Poole 2.3):* Any underdetermined $m \times n$ homogeneous linear system ($n > m$) has infinitely many solutions.